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# Sequences with equi-distributed extreme points in uniform polynomial approximation

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**Abstract**

Let $E$ be a compact set in $\mathbb{C}$ with connected complement and positive logarithmic capacity. For any $f$ continuous on $E$ and analytic in the interior of $E$, we consider the distribution of extreme points of the error of best uniform polynomial approximation on $E$. Let $\Lambda = (n_j)$ be a subsequence of $\mathbb{N}$ such that $n_{j+1}/n_j \to 1$. If, for $n \in \Lambda$, $A_n(f) \subseteq \partial E$ denotes the set of extreme points of the error function, we prove that there is a subsequence $\Lambda'$ of $\Lambda$ such that the distribution of any $(n+2)$th Fekete point set $\mathscr{F}_{n+2}$ of $A_n(f)$ tends weakly to the equilibrium distribution on $E$ as $n \to \infty$ in $\Lambda'$. Furthermore, we prove a discrepancy result for the distribution of the point sets $\mathscr{F}_{n+2}$ if the boundary of $E$ is smooth enough.


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## 1. Introduction

The problem we are considering started with a paper of Kadec [6]. He proved that for any real function $f \in C[-1, 1]$ the alternation points of the best approximation

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with respect to algebraic polynomials of degree $n \in \mathbb{N}$ are distributed like the extreme points of the Chebychev polynomials, at least for a subsequence. He even gave an estimate of the discrepancy between the two distributions in terms of the degree $n \in \mathbb{N}$.

The theorem of Kadec has been extended to polynomial approximation on compact sets $E \subseteq \mathbb{C}$ by Blatt et al. [4]. In this case, one has to choose an appropriate subset of the extremal point set. For a subsequence, this sequence of subsets is distributed like the equilibrium distribution of $E$, the set, where the approximation takes place.

Of course, it is natural to ask about the nature of any subsequence for which the extreme points of the best approximation can behave badly. Lorentz proved in [11] for the interval case that the convergence to the expected distribution may not take place for all subsequences. This result was generalized by Kroó and Saff [9] to the complex case.

Kroó [7] proved for $1 < p \leqslant \infty$, $f \in C[-1,1]$ that if the error function of best $L^p$-approximation has no zero in some subinterval $(a,b) \subset [-1,1]$ for some subsequence $(n_j)_{j\in\mathbb{N}}$, then

$$\limsup_{j\to\infty} \frac{n_{j+1}}{n_j} > 1.$$

This was an improvement of a weaker theorem by Kroó and Swetits [10].

In [2], this result was sharpened. Let $1 < p \leqslant \infty$, $f \in C[-1,1]$ and $(n_j)_{j\in\mathbb{N}}$ a subsequence, such that

$$\lim_{j\to\infty} \frac{n_{j+1}}{n_j} = 1.$$

Then there exist $n_j + 2$ points in $[-1,1]$, where the error function of the best $L^p$-approximation has alternating signs and the point measures $\tau_{n_j}$ associated with these points of alternation have the equilibrium distribution on $[-1,1]$ as a weak limit point. An analogous result for $L^1$-approximation was proved in [3], compare also [8].

The outline of this paper is as follows. In Section 2, we state the distribution results of extremal point sets for complex approximation by polynomials on compact sets $E \subset \mathbb{C}$. In Section 3, we compare the error of best approximation on discrete point sets of $E$ which consists, up to a simple point, of a Fekete point set of $E$ with the global error on $E$ (Lemma 2). In Lemmas 3 and 4, we derive upper bounds for the ratio of the discrete error on $\{x_i\}_{i=0}^{n+1} \subset E$ to the global error on $E$ in terms of the derivative of $\omega_n(x) = \prod_{i=0}^{n+1}(x - x_i)$ at the zeros $x_i$, $0 \leqslant i \leqslant n+1$. Section 4 describes a distribution result for the zeros of the monic polynomials $\omega_n$ based on the derivative at the zeros. Section 5 provides the proof of the theorems using the auxiliary lemmas of the previous sections.

## 2. Main results

We consider a compact set $E \subset \mathbb{C}$ with positive capacity, such that the complement $\Omega := \bar{\mathbb{C}} \backslash E$ is connected; i.e., there is a Green function $G = G_E$ on $\Omega$, which converges to 0 quasi-everywhere on $\partial E$. It is well known that in this case

$$G(z) = -U^{\mu}(z) - \log \operatorname{cap} E, \quad z \in \Omega,$$

where $\operatorname{cap} E$ denotes the capacity of $E$, $\mu = \mu_E$ is the equilibrium distribution of $E$ supported on $\partial E$ and, for unit measures $v$ on $E$,

$$U^{v}(z) = \int \log\left(\frac{1}{|z-t|}\right) dv(t)$$

is the logarithmic potential of $v$.

By

$$p_n^*(f) = p_{n,E}^*(f)$$

we denote the best approximating polynomial to a function $f$, continuous on $E$ and analytic in the interior of $E$, with respect to $\mathscr{P}_n$, the set of all complex polynomials of degree $n$; i.e.,

$$e_n(f) = e_{n,E}(f) := ||f - p_n^*(f)||_E = \min_{p \in \mathscr{P}_n} ||f - p||_E,$$

where the norm is the uniform norm on $E$. We denote the extremal set of best approximation by

$$\mathscr{A}_n = \mathscr{A}_n(f) = \{z \in E : |f(z) - p_n^*(f)(z)| = e_n(f)\}.$$

It is well known that

$$e_{n,\mathscr{A}_n}(f) = e_n(f).$$

$\mathscr{A}_n(f)$ is a subset of $\partial E$, which has at least $n+2$ points and can be an infinite set.

For compact $M \subset \mathbb{C}$, denote by

$$\mathscr{F}_n(M) = \{x_1, \ldots, x_n\} \subseteq M,$$

a Fekete point set of $n$ points of $M$; i.e., a set of points in $M$ such that

$$\prod_{i \neq j, 1 \leqslant i,j \leqslant n} |x_i - x_j|$$

is maximal.

If $B \subset \mathbb{C}$ is a finite point set we denote by $\tau(B)$ the normalized counting measure of $B$, i.e., if $B$ consists of exactly $m$ points then $\tau$ associates the mass $1/m$ to each point.

Fekete [4] in subsets $\mathscr{F}_{n+2}(\mathscr{A}_n(f))$ were chosen in $\mathscr{A}_n(f)$ for each $n \in \mathbb{N}$. For these point sets it was shown that there exists a subsequence $\Lambda' \subset \mathbb{N}$ such that

$$\tau(\mathscr{F}_{n+2}(\mathscr{A}_n(f))) \xrightarrow{*} \mu_E \quad \text{as } n \to \infty, \ n \in \Lambda', \tag{1}$$

where "$\xrightarrow{*}$" denotes weak$^*$ convergence of the measures.

In this paper, we want to characterize subsequences $\Lambda \subseteq \mathbb{N}$, such that this weak* convergence (1) holds for some subsequence $\Lambda'$ of $\Lambda$. Our main result is in the following theorem.

**Theorem 1.** *Assume that* $\Lambda = (n_j)_{j \in \mathbb{N}}$ *is a subsequence of* $\mathbb{N}$ *with*

$$\lim_{j \to \infty} \frac{n_{j+1}}{n_j} = 1.$$

*Then we can choose a subsequence* $\Lambda' \subseteq \Lambda$ *such that*

$$\tau(\mathscr{F}_{n+2}(\mathscr{A}_n(f))) \xrightarrow{*} \mu_E \quad \text{as } n \to \infty, \ n \in \Lambda'. \tag{2}$$

Thus, if a subsequence $(n_j)_{j \in \mathbb{N}}$ of $\mathbb{N}$ has the property that no asymptotically equal distributed points can be chosen in $\mathscr{A}_{n_j}(f)$ (not even for a subsequence), then it must contain gaps; i.e.

$$\limsup_{j \to \infty, j \in \Lambda} \frac{n_{j+1}}{n_j} > 1.$$

For a set $E$ that is either a Jordan curve or an arc, we can define the discrepancy of two measures $\nu_1$ and $\nu_2$ supported on $E$ by

$$D[\nu_1 - \nu_2] = \sup\{|(\nu_1 - \nu_2)(J)| : J \text{ is a subarc of } \partial E\}.$$

For such sufficiently smooth Jordan arcs and curves it is possible to sharpen the weak* convergence of Theorem 1 to a discrepancy estimate between the measures $\tau(\mathscr{F}_{n+2}(\mathscr{A}_n(f)))$ and $\mu_E$.

**Theorem 2.** *Let* $L = \partial E$ *be a piecewise Dini-smooth Jordan curve with all inner angles less than or equal to* $\pi$, *or let* $L$ *be a piecewise Dini-smooth Jordan arc. Then*

$$D[\tau(\mathscr{F}_{n+2}(\mathscr{A}_n(f))) - \mu_E] \leqslant C\sqrt{\frac{k}{n}\log\left(1 + \frac{n}{k}\right) + \frac{1}{n}\log\left(\frac{e_n(f)}{e_n(f) - e_{n+k}(f)}\right)} \tag{3}$$

*for all* $1 \leqslant k \leqslant n$, *where* $C$ *is a positive constant independent of* $f$, $n$ *and* $k$.

Now, Theorem 2 allows us to formulate the asymptotic result in Theorem 1 in terms of discrepancy.

**Theorem 3.** *Let* $E$ *be as in Theorem* 2 *and let* $\Lambda = (n_j)_{j \in \mathbb{N}}$ *a subsequence of* $\mathbb{N}$ *with*

$$\lim_{j \to \infty} \frac{n_{j+1}}{n_j} = 1.$$

*Then there exists a subsequence* $\Lambda' \subseteq \Lambda$, *such that*

$$D[\tau(\mathscr{F}_{n_j+2}(\mathscr{A}_{n_j}(f))) - \mu_E] \leqslant C\sqrt{\frac{\log \gamma_j}{\gamma_j}} \tag{4}$$

*for all* $n_j \in \Lambda'$, *where*

$$\gamma_j = \frac{n_j}{n_{j+1} - n_j}$$

*and C is a positive constant, independent of* $f$.

Note that by the hypothesis of Theorem 3 $\lim_{j\to\infty} \gamma_j = \infty$.

**Remark.** In all theorems above the convergence results hold also for a Fekete point set

$$\mathscr{F}_{n+2-k}(\mathscr{A}_n(f))$$

instead of $\mathscr{F}_{n+2}(\mathscr{A}_n(f))$ where the number $k \in \mathbb{N}$ is fixed.

## 3. Interrelations between global and discrete approximations

One of our tools in the proofs is the relation between the norm of polynomials on compact sets and their discrete norms on Fekete point sets. We use a lemma of Walsh.

**Lemma 1.** *For any compact set* $M \subset \mathbb{C}$ *with at least* $n+1$ *points choose a Fekete point set* $\mathscr{F}_{n+1}(M)$ *consisting of* $n+1$ *points. Then for any polynomial* $p \in \mathscr{P}_n$, *we have*

$$||p||_M \leqslant (n+1)||p||_{\mathscr{F}_{n+1}(M)}.$$

For the proof see Section 7.9, Lemma, p. 177, of Walsh [13].

With this lemma we are able to obtain lower bounds for the minimal error in polynomial approximation on special discrete sets in terms of the global minimal error on compact sets $M \subset \mathbb{C}$.

**Lemma 2.** *Let* $g$ *be continuous on the compact set* $M \subset \mathbb{C}$ *consisting of more than* $m$ *points where* $m \geqslant n+2$. *Then, for any Fekete point set* $\mathscr{F}_{m-1}(M)$ *there exists a point* $x_g \in M$ *such that*

$$e_{n,A}(g) \geqslant \frac{1}{2m-1} e_{n,M}(g),$$

*where*

$$A = \mathscr{F}_{m-1}(M) \cup \{x_g\}.$$

Of course, one cannot choose one and the same $A$ for all $g$. However, only one point in $A$ depends on $g$ while the other points depend on $M$ only.

**Proof.** Choose a Fekete point set $F = \mathscr{F}_{m-1}(M)$ in $M$. Then define for each $x \in M$

$$e(x) = e_{n,F\cup\{x\}}(g).$$

It is easy to see that $e(x)$ depends continuously on $x$. Thus there is an $x_g \in M$, which maximizes $e(x)$. We set

$$A = F \cup \{x_g\}.$$

Let us denote by $p_x \in \mathscr{P}_n$ the polynomial of best approximation to $g$ on $F \cup \{x\}$, i.e.,

$$e(x) = ||g - p_x||_{F \cup \{x\}}.$$

Then for all $t \in M$, using the previous lemma for polynomials in $\mathscr{P}_{m-2}$,

$$\begin{aligned} |g(t) - p_{x_g}(t)| &\leqslant |g(t) - p_t(t)| + |p_t(t) - p_{x_g}(t)| \\ &\leqslant e(t) + (m-1)||p_t - p_{x_g}||_F \\ &\leqslant e(t) + (m-1)(||p_t - g||_F + ||g - p_{x_g}||_F) \\ &\leqslant e(t) + (m-1)(e(t) + e(x_g)) \\ &\leqslant (2m-1)e(x_g). \end{aligned}$$

Hence,

$$e_{n,M}(g) \leqslant ||g - p_{x_g}||_M \leqslant (2m-1)e(x_g).$$

This finishes the proof. $\square$

On the other hand, we also need lower bounds for the discrete minimal error on point sets of $n+2$ points. For that reason, we define for $r > 1$ the level line

$$\Gamma_r := \{z \in \Omega : G(z) = \log r\}$$

of Green's function $G(z)$. Moreover, for $r > 1$ let

$$E_r = E \cup \{z \in \mathbb{C} : G(z) \leqslant \log r\}.$$

Then there exists $r_E > 1$ such that $\Gamma_r$ is an analytic and simple closed curve for all $r > r_E$.

In the following lemmas we fix $n+2$ (pairwise disjoint) points

$$x_0, x_1, \ldots, x_{n+1} \in E$$

and define

$$\omega_n(x) := \prod_{i=0}^{n+1} (x - x_i).$$

**Lemma 3.** *Let $g$ be a function analytic on $E_r$, $g \notin \mathscr{P}_n$, where*

$$r > \max(r_E, 2 \operatorname{diam} E / \operatorname{cap} E).$$

*Then there exists a positive constant $C > 0$, depending only on $E$, such that*

$$\frac{e_{n,\{x_0,\ldots,x_{n+1}\}}(g)}{e_{n,E_r}(g)} \leqslant C \frac{\alpha_E^{n/r}}{(r \operatorname{cap} E)^{n+1}} \min_{0 \leqslant i \leqslant n+1} |\omega_n'(x_i)|,$$

*where*

$$\alpha_E = e^{2\,\mathrm{diam}\,E/\mathrm{cap}\,E}$$

*and* diam $E$ *denotes the diameter of* $E$.

**Proof.** We denote by $P_n(g) \in \mathscr{P}_n$ the interpolating polynomial of degree at most $n$ to $g$ in $x_1, \dots, x_{n+1}$. Then

$$e_{n,\{x_0,\dots,x_{n+1}\}}(g) \leqslant ||g - P_n(g)||_{\{x_0,\dots,x_{n+1}\}} = |g(x_0) - P_n(g)(x_0)|. \tag{5}$$

By the Hermite formula we have for any polynomial $q_n \in \mathscr{P}_n$

$$g(x_0) - P_n(g)(x_0) = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{\tilde{\omega}_n(x_0) g(z)}{\tilde{\omega}_n(z)(z - x_0)}\, dz, = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{\tilde{\omega}_n(x_0)(g - q_n)(z)}{\tilde{\omega}_n(z)(z - x_0)}\, dz, \tag{6}$$

where

$$\tilde{\omega}_n(z) = (z - x_1)\cdot \dots \cdot (z - x_{n+1}).$$

Note that

$$\tilde{\omega}_n(x_0) = \omega_n'(x_0).$$

Inserting the best approximation $q_n = p^*_{n,E_r}(g)$ to $g$ on $E_r$ into (6), we get by (5)

$$e_{n,\{x_0,\dots,x_{n+1}\}}(g) \leqslant \frac{C_0 |\omega_n'(x_0)| e_{n,E_r}(g) \mathrm{length}(\Gamma_r)}{\delta_r^{n+2}} \tag{7}$$

with a constant $C_0 > 0$ and

$$\delta_r = \mathrm{dist}(\Gamma_r, E).$$

Since

$$U^{\mu_E}(z) = -\int_E \log|z - t|\, d\mu_E(t)$$

we obtain for $z \in \Gamma_r$

$$-U^{\mu_E}(z) = \log r + \log \mathrm{cap}\, E \leqslant \log(\delta_r + \mathrm{diam}\, E)$$

or

$$r\,\mathrm{cap}\, E \leqslant \delta_r + \mathrm{diam}\, E.$$

Hence,

$$\delta_r \geqslant r\,\mathrm{cap}\, E - \mathrm{diam}\, E = r\,\mathrm{cap}\, E\left(1 - \frac{\mathrm{diam}\, E}{r\,\mathrm{cap}\, E}\right)$$

and for $r \geqslant 2\,\mathrm{diam}\, E/\mathrm{cap}\, E$

$$\delta_r \geqslant r\alpha_E^{-1/r}\,\mathrm{cap}\, E. \tag{8}$$

The complement of the disc $\{z : |z| \leqslant r\}$ is mapped to the exterior of $\Gamma_r$ by the Riemann mapping with

$$\Phi(\infty) = \infty, \quad \Phi'(\infty) = \operatorname{cap} E.$$

Using the integral formula for curve length, we get

$$\operatorname{length}(\Gamma_r) = \int_0^{2\pi} \left| \frac{d}{dt} \Phi(e^{it}) \right| dt \leqslant C_1 r \tag{9}$$

for all $r > r_E$ with some fixed constant $C_1'$, independent of $r$.

Inserting (8) and (9) into (7) we obtain

$$e_{n,\{x_0,\dots,x_{n+1}\}}(g) \leqslant \frac{C_2 |\omega_n'(x_0)| e_{n,E_r}(g) \alpha_E^{n/r}}{(r \operatorname{cap} E)^{n+1}}$$

with some constant $C_2$, independent of $r$ and $n$.

Of course, we can repeat the same proof with $x_i$, $i = 1, \dots, n+1$, instead of $x_0$. This yields the lemma. □

**Lemma 4.** *Let $n \in \mathbb{N}$ and $x_0, \dots, x_{n+1} \in E$. For all $1 \leqslant k \leqslant n$ and $P_{n+k} \in \mathscr{P}_{n+k} \backslash \mathscr{P}_n$ we have*

$$\frac{e_{n,\{x_0,\dots,x_{n+1}\}}(P_{n+k})}{e_n(P_{n+k})} \leqslant C \left( \frac{\beta n}{k} \right)^k \frac{1}{(\operatorname{cap} E)^{n+1}} \min_{0 \leqslant i \leqslant n+1} |\omega_n'(x_i)|,$$

*where $\beta$ and $C$ are positive constants depending only on the geometry of $E$.*

**Proof.** We apply the previous lemma to $g = P_{n+k}$. By the Bernstein–Walsh inequality, we have

$$e_{n,E_r}(P_{n+k}) \leqslant \| P_{n+k} - p_n^*(P_{n+k}) \|_{E_r} \leqslant r^{n+k} e_n(P_{n+k}). \tag{10}$$

Here, $p_n^*(P_{n+k})$ refers to the best approximation of $P_{n+k}$ on $E$.

Next we choose

$$r_n = \frac{n \log(\alpha_E)}{k},$$

then we get

$$r_n = 2 \frac{n}{k} \frac{\operatorname{diam} E}{\operatorname{cap} E} \geqslant 2 \frac{\operatorname{diam} E}{\operatorname{cap} E}.$$

Moreover, we have

$$\alpha_E^{n/r_n} = \alpha_E^{k/\log \alpha_E} = e^k.$$

Finally, we set

$$r := \max(r_n, r_E).$$

Then

$$\alpha_E^{n/r} \leqslant \alpha_E^{n/r_n} \leqslant e^k$$

and

$$r \leqslant \max\left(r_E, \frac{n \log \alpha_E}{k}\right)$$
$$\leqslant \max\left(r_E, 2n\frac{\operatorname{diam} E}{\operatorname{cap} E}\right).$$

Inserting this together with (10) into the inequality of Lemma 3, we finally obtain Lemma 4 with

$$\beta = \max(r_E, 2e \operatorname{diam} E/\operatorname{cap} E). \qquad \square$$

## 4. Weak* convergence of discrete point sets

In Lemma 3 resp. Lemma 4 we have obtained lower bounds for the expression

$$\min_{0 \leqslant i \leqslant n+1} \frac{|\omega_n'(x_i)|}{(\operatorname{cap} E)^{n+1}}, \quad \text{where } \omega_n(x) = \prod_{i=0}^{n+1} (x - x_i).$$

Such lower bounds have been used by Hüsing [5], and Blatt and Andrievskii [1] in the complex plane to derive discrepancy results of the difference of the measure $\mu_E$ and the normalized counting measure of the point set $\{x_i\}_{i=0}^{n+1}$.

But if one is only interested in weak* convergence the following lemma is very useful.

**Lemma 5.** *Let*

$$\omega_n(z) = (z - x_{1,n}) \cdot \ldots \cdot (z - x_{n,n})$$

*be a sequence of monic polynomials such that all zeros $x_{i,j}$ are on $\partial E$, defined for $n \in \Lambda \subset \mathbb{N}$ with*

$$\liminf_{\substack{n \to \infty \\ n \in \Lambda}} \min_{1 \leqslant i \leqslant n} \frac{|\omega_n'(x_{i,n})|^{1/n}}{\operatorname{cap} E} \geqslant 1.$$

*Then $\tau_n$ converges weakly to $\mu_E$ as $n \in \Lambda \to \infty$, where $\tau_n = \tau_n(\omega_n)$ is the normalized zero counting measure of $\omega_n$.*

**Proof.** We may assume that $x_{1,n}, \ldots, x_{n,n}$ are pairwise disjoint. Interpolating the polynomial 1, we get for $x \notin E$

$$1 = \sum_{i=1}^{n} \frac{\omega_n(x)}{\omega_n'(x_{i,n})(x - x_{i,n})}.$$

Thus

$$|\omega_n(x)| \geqslant \frac{1}{n} \operatorname{dist}(x, E) \min_{1 \leqslant i \leqslant n} |\omega_n'(x_{i,n})|.$$

By our hypothesis, we get

$$\liminf_{\substack{n\to\infty\\ n\in\Lambda}} |\omega_n(x)|^{1/n} \geqslant \operatorname{cap} E \tag{11}$$

for all $x\notin E$. Now let $\tau$ be any weak limit point of the sequence of measures $(\tau_n)_{n\in\Lambda}$. Then

$$\frac{1}{n}\log|\omega_n(x)| = -U^{\tau_n}(x),$$

and therefore

$$\lim_{\substack{n\to\infty\\ n\in\Lambda}} \frac{1}{n}\log|\omega_n(x)| = -U^{\tau}(x).$$

for all $x\notin E$. From (11), we get

$$U^{\tau}(x)\leqslant -\log\operatorname{cap} E, \quad x\notin E.$$

Now, $U^{\tau} - U^{\mu_E}$ is a harmonic function in $\Omega = \mathbb{C}\backslash E$ which has the value 0 at the point $\infty$. Since $U^{\mu_E}(z) = -\log\operatorname{cap} E$ quasi-everywhere on $E$,

$$\limsup_{x\to\zeta}(U^{\tau}(x) - U^{\mu_E}(x))\leqslant 0$$

for quasi-every $\zeta\in\partial\Omega = \partial E$. By the generalized maximum principle, $U^{\tau} = U^{\mu_E}$ on $\Omega$, and $\tau$ is supported in $\partial E$. It is known that this implies $\tau = \mu_E$. □

Note, that the lim inf inferior in Lemma 5 can never be larger than 1 [12, Theorem III. 1.7].

## 5. Proofs of the theorems

We are now in position to prove the theorems.

**Proof of Theorem 1.** Since the telescoping product

$$\prod_{j=1}^{\infty}\frac{e_{n_{j+1}}(f)}{e_{n_j}(f)}$$

converges to 0, the series

$$\sum_{j=1}^{\infty}\left(\frac{e_{n_{j+1}}(f)}{e_{n_j}(f)} - 1\right)$$

must diverge, and we get for a subsequence $\Lambda_0\subset\mathbb{N}$

$$\frac{e_{n_{j+1}(f)} - e_{n_j}(f)}{e_{n_j}(f)}\geqslant\frac{1}{j^2}\geqslant\frac{1}{n_j^2}, \quad j\in\Lambda_0. \tag{12}$$

Now, we want to apply Lemma 4 with $n = n_j$ and $k = n_{j+1} - n_j$ and

$$P_{n+k} := q_j = p^*_{n_j} - p^*_{n_{j+1}} \in \mathscr{P}_{n_{j+1}}.$$

We need a subset of $n_j + 2$ points of $\mathscr{A}_{n_j}(f)$, an estimate from below for the minimal error of $P_{n+k}$ on this subset, and an estimate from above for the global error on $E$.

Clearly

$$e_{n_j}(q_j) \leqslant \|p^*_{n_j}(f) - p^*_{n_{j+1}}(f)\|_E \leqslant e_{n_j}(f) + e_{n_{j+1}}(f) \leqslant 2e_{n_j}(f) \tag{13}$$

which provides the needed estimate from above.

To obtain an estimate from below we use

$$\begin{aligned} e_{n_j,\mathscr{A}_{n_j}(f)}(q_j) &= e_{n_j,\mathscr{A}_{n_j}(f)}(p^*_{n_{j+1}}(f)) \\ &\geqslant e_{n_j,\mathscr{A}_{n_j}(f)}(f) - e_{n_{j+1}}(f) \\ &= e_{n_j}(f) - e_{n_{j+1}}(f). \end{aligned} \tag{14}$$

If the set $\mathscr{A}_{n_j}(f)$ consists of more than $n_j + 2$ points then we have to reduce the set $\mathscr{A}_{n_j}(f)$ to $n_j + 2$ points. We arrange this by applying Lemma 2 twice. With $m = n_j + 3$ we get a point $\tilde{z}_j \in A_{n_j}(f)$ such that

$$e_{n_j,F}(q_j) \geqslant \frac{1}{2n_j + 5}(e_{n_j}(f) - e_{n_{j+1}}(f))$$

with

$$F = \mathscr{F}_{n_j+2}(A_{n_j}(f)) \cup \{\tilde{z}_j\}.$$

With $m = n_j + 2$, we get a point $z_j \in F$ such that

$$e_{n_j,B_j}(q_j) \geqslant \frac{1}{(2n_j + 3)(2n_j + 5)}(e_{n_j}(f) - e_{n_{j+1}}(f)), \tag{15}$$

where

$$B_j := \{x_{0,j}, \ldots, x_{n_j+1,j}\} = \mathscr{F}_{n_j+1}(F) \cup \{z_j\}.$$

Defining

$$\omega_j(x) = (x - x_{0,j}) \cdots (x - x_{n_j+1,j})$$

we obtain with (12)–(15) for $j \in \Lambda_0$ that

$$\frac{e_{n_j,B_j}(q_j)}{e_n(q_j)} \geqslant \frac{C_1}{n_j^4},$$

where $C_1$ is a positive constant, independent of $n$ and $f$. Hence, by Lemma 4

$$\min_{0 \leqslant i \leqslant n_j+1} \frac{|\omega_j'(x_{i,j})|}{(\operatorname{cap} E)^{n_j+2}} \geqslant \frac{C}{n_j^4}\left(\frac{n_{j+1} - n_j}{\beta n_j}\right)^{n_{j+1}-n_j} \tag{16}$$

with some constant $C > 0$.

In the case that $\mathscr{A}_{n_j}(f)$ has exactly $n_j + 2$ points, let

$$B_j := \mathscr{A}_{n_j}(f) = \{x_{0,j}, \ldots, x_{n_j+1,j}\}.$$

Using (12)–(14) we get

$$\frac{e_{n_j,B_j}(q_j)}{e_n(q_j)} \geqslant \frac{C_2}{n_j^2}$$

with some constant $C_2 > 0$. Therefore, again by Lemma 4 we obtain inequality (16).

Now we can apply Lemma 5. We have to show that the $n_j$th root of the right-hand side of (16) tends to 1. However, by the assumptions of Theorem 1

$$\alpha_j = \frac{n_{j+1} - n_j}{n_j} \to 0 \quad \text{as } j \to \infty$$

and thus

$$(\alpha_j/\beta)^{\alpha_j} \to 1 \quad \text{as } j \to \infty.$$

Therefore,

$$\liminf_{\substack{j \to \infty \\ j \in \Lambda_0}} \min_{0 \leqslant i \leqslant n_j+1} \frac{|\omega_j'(x_{i,j})|^{1/(n_j+2)}}{\operatorname{cap} E} \geqslant 1$$

and, by Lemma 5, the normalized counting measures $\tau(B_j)$ converge weakly to $\mu_E$ as $j \to \infty$, $j \in \Lambda_0$.

Since the sets

$$B_j = \{x_{0,j}, \ldots, x_{n_j+1,j}\},$$

and the sets

$$\mathscr{F}_{n_j+2}(\mathscr{A}_{n_j}(f))$$

each have at least $n_j + 1$ points in common, we get the same distribution for both sequences of sets.

This finishes the proof of Theorem 1. $\square$

**Proof of Theorem 2.** We apply Lemma 4 with $P_{n+k} = p_{n+k}^*(f)$. Using the same reasoning as in the proof of Theorem 1, we know that there exist $n+1$ points $x_i \in \mathscr{F}_{n+2}(\mathscr{A}_n(f))$, $1 \leqslant i \leqslant n+1$, and an additional point $x_0 \in \mathscr{A}_n(f)$ such that

$$\min_{0 \leqslant i \leqslant n+1} \frac{|\omega_n'(x_i)|}{(\operatorname{cap} E)^{n+2}} \geqslant C_1 \frac{e_n(f) - e_{n+k}(f)}{n^2 e_n(f)} \left(\frac{k}{\beta n}\right)^k,$$

with some positive constants $C_1$ and $\beta$ depending on $E$ only.

Now, we apply the discrepancy theorem in [1, Theorem 4.15]. If

$$\min_{0 \leqslant i \leqslant n+1} \frac{|\omega_n'(x_i)|}{(\operatorname{cap} E)^{n+2}} \geqslant \frac{1}{A_n}, \quad n \leqslant A_n < e^n,$$

then this theorem states that there is a positive constant $c$ depending on $E$ only such that

$$D[\tau_n - \mu_E] \leqslant c\sqrt{\frac{\log A_n}{n}},$$

where $\tau_n = \tau(\omega_n)$ is the normalized zero counting measure of $\omega_n$.

For $\beta \geqslant 1$ we use

$$\frac{\beta n}{k} \leqslant 1 + \frac{\beta n}{k} \leqslant \left(1 + \frac{n}{k}\right)^{\beta}.$$

If $\beta < 1$, we just replace it by $\beta = 1$.

Hence, inequality (3) is a direct consequence. □

**Proof of Theorem 3.** Let

$$k = n_{j+1} - n_j, \quad n = n_j.$$

We choose again a subsequence $\Lambda_0 \subseteq \mathbb{N}$ such that (12) holds. Then

$$\Lambda' := \{n_j : j \in \Lambda_0\}$$

satisfies inequality (4) of Theorem 3. □

Inspecting the proofs of the theorems, the final remark in Section 2 is quite obvious. Moreover, it turns out that the proofs are even simpler if $k \geqslant 1$ in $\mathscr{F}_{n+2-k}(\mathscr{A}_n(f))$ since it is not necessary to apply Lemma 4 twice.

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